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On Perturbations of Positive c_0(Semi)Groups on Banach Lattices and Applications

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We study the existence of wave operators, in terms of limiting absorption principles for a class positively perturbed positive (semi)groups on Banach lattices. This work is a partial extension of the L^1 scattering theory by M. Mokhtar-Kharroubi (*J. Funct. Anal.* **115**, 1993, 119–145). An application to perturbed heat semigroups is given. © 1996 Academic Press, Inc.

I. INTRODUCTION

The present paper is a *partial* extension to Banach lattices of an abstract theory of scattering for positive groups on L^1 spaces by one of the authors [1].

Actually the present work (as well as [1]) was motivated by transport theory problems. Usually the relevant physical spaces in transport theory are L^1 spaces where *positivity* plays also an important role. On the other hand and from the mathematical point of view L^1 spaces enjoy the crucial property that *the norm is additive on the positive cone*. This nice property plays a basic role in the L^1 scattering theory and its applications to transport equations [1]. We refer the reader to [1] for the literature on scattering transport theory. This literature is mainly concerned with neutron transport like equations on L^1 *spaces*.

The extension to L^p spaces $(1 < p < +\infty)$ of the scattering transport theory meets several mathematical difficulties. We note that there exists an abstract scattering theory on Banach lattices by Umeda [2]. Unfortunately Umeda's theory does not apply to transport equations on L^p spaces $(1 < p < +\infty)$ because the basic assumptions are not fulfilled when $p > 1$.

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Actually the more convenient mathematical tool for treating scattering neutron transport problems in L^p spaces $(1 < p < +\infty)$ is the *factorization technique* used, for instance, by Kato [3] and Lin [4] (see [5]).

On the other hand the transport theory of particle swarms [6] has the peculiarity that the perturbation is *compact* in contrast to the neutron transport equation. The equation has the simpler form

$$\frac{\partial f}{\partial t} + a \cdot \frac{\partial f}{\partial v} + \sigma(v) f(v,t) = \int_{\mathbb{R}^N} B(v,v') f(v',t)\, dv' = Bf. \qquad (1)$$

The investigation of the wave operators for (1) [i.e., the asymptotic equivalence of the dynamics of (1) and the free dynamics $(B = 0)$] is given in [6] in L^1 setting. Time dependent operators are also considered in [7, 8].

The present work was motivated by a scattering theory of (1) on L^p spaces $(1 < p < +\infty)$ by exploiting the *compactness* of the perturbation B. Since compact operators (on L^p spaces) are limits of finite rank operators, our idea was to build a scattering theory on *Banach lattices* for finite rank perturbations with the hope to recover (by some limiting process) the compact perturbations or, at least, the nuclear ones. It turns out that *the* L^1 *techniques* of [1] *extend* (partially) to *finite rank perturbations on Banach lattices* under suitable positivity assumptions.

Indeed we are able to recover a good part of the *abstract* results of [1] by relying on the existence of wave operators to *limiting absorption principles*. Unfortunately our results do not apply to (1) because the limiting absorption principle fails! It turns out that the convenient approach of (1) in the L^p setting $(1 < p < +\infty)$ is *still* the factorization technique [9].

However, the functional analytic results we obtain seem interesting on their own. On the other hand, as by-products, we obtain *criteria of boundedness* for positively perturbed positive (semi)groups on Banach lattices when the perturbation is of finite rank or even in a suitable class of nuclear perturbations. Applications of such criteria to *perturbed heat semigroups* on $\mathbb{R}^N$ are given as an illustration of our techniques.

It is worth noting that one rank (or nuclear) perturbation techniques are well known in *quantum scattering theory* (see [10, Chap. X]). The techniques we use are however *completely different* since we deal with *positive* groups (in the lattice sense) on Banach lattices. Moreover we rely on the existence of wave operators to a limiting absorption principle which is itself *intimately connected to a monotone convergence theorem* (in the lattice sense). Actually our approach relies on the *lattice structure* of the underlying space and, as such, is very different from the Hilbert space theory [10, Chap. X; 11].

In rough terms we obtain particularly the following results: Let E be a Banach lattice with order continuous norm (e.g., L^p spaces $p \in [1, \infty)$) and let $\{S(t); t \in \mathbb{R}\}$ be a positive and bounded (c_0)-group on E with generator T. Let $B \in \mathscr{L}(D(T); E)$ be with rank one, i.e., $Bx = \varphi(x)b$ where $\varphi \in D(T)'$ and $b \in E$. The *crucial* assumption is that the linear form φ be positive, i.e., $\varphi(x) \geq 0$ if $x \in D(T)$, $x \geq 0$.

Then one proves that *the existence of a bounded* c_0*(semi)group* with generator $T + B$ as well as the existence of the wave operators

$$s \lim_{t \to +\infty} e^{t(T+B)} e^{-tT}, \qquad s \lim_{t \to -\infty} e^{t(T+B)} e^{-tT}, \qquad s \lim_{t \to +\infty} e^{-tT} e^{t(T+B)}$$

relies on the existence of the strong limits

$$B(0_+ - T)^{-1} = s \lim_{\lambda \to 0_+} B(\lambda - T)^{-1},$$

$$B(0_- - T)^{-1} = s \lim_{\lambda \to 0_-} B(\lambda - T)^{-1}$$

and on *the size of their spectral radii*.

We obtain similar results when $B = \sum_{i=0}^{\infty} B_i$ where the B_i are positive one rank operators. We exemplify some of our results by considering the heat semigroup on $L^p(\mathbb{R}^N)$, $p \in [1, \infty)$.

A *positive* linear form on the domain of the laplacian is actually a positive Radon measure $d\mu$. Then we prove that $B(0_+ - T)^{-1}$ exists if and only if the Riez potential of $d\mu$ belongs to L^{p^*} $(1/p + 1/p^* = 1)$ and *compute exactly* $r\sigma(B(0_+ - T)^{-1})$ in terms of $d\mu$. Neighbouring results are also given. Finally it turns out that the limiting absorption principle (i.e., the existence of $s \lim_{\lambda \to 0_+} B(\lambda - T)^{-1}$) fails when $N = 1$ or 2. Since our conditions are only *sufficient* the boundedness of $e^{t(\Delta + B)}$ is open for $N = 1, 2$. On the other hand those conditions are *necessary* and sufficient on L^1 spaces [1] and consequently $\{e^{t(\Delta+B)}, t \geq 0\}$ *is not* bounded on $L^1(\mathbb{R}^N)$ whatever the size of $d\mu$.

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II. ON A CLASS OF PERTURBATIONS OF POSITIVE SEMIGROUPS ON BANACH LATTICES

We first consider positive one rank perturbations. We will show at the end of the section how the results extend to a class of nuclear perturbations.

Let E be a Banach lattice (with order continuous norm) and let $\{S(t); t \in \mathbb{R}\}$ be a *positive* and *bounded* (c_0)*-group* of linear operators on E

with generator T. Let $B \in \mathscr{L}_+(D(T); E)$ be a positive one rank operator given by $Bx = \varphi(x)b$ for $x \in D(T)$ with φ a *positive* linear form on $D(T)$ and $b \in E_+$.

Remark 1. The *positiveness* of φ is *crucial* in our analysis. On the other hand, the positiveness of b is not essential.

The aim of this section is to give sufficient conditions on B under which $T + B$ generates a *bounded* (semi)group. This result is provided by the following

THEOREM 1. (1) *Let* $B(0_+ - T)^{-1} := s\lim_{\lambda \to 0_+} B(\lambda - T)^{-1}$ *exist and let* $r_\sigma[B(0_+ - T)^{-1}] < 1$, *then* $T + B$ *is the generator of a positive and* bounded (c_0)*-semigroup* $\{V(t); t \geq 0\}$. *Moreover, for all* $R \in \mathscr{L}(E)$, $s\lim_{t \to +\infty} RV(t)$ *exists if and only if* $s\lim_{t \to +\infty} RS(t)$ *exists. These limits are then related as*

$$[RV(+\infty)] = [RS(+\infty)]\left[I - B(0_+ - T)^{-1}\right]^{-1}.$$

(2) *If, in addition,* $B(0_- - T)^{-1} := s\lim_{\lambda \to 0_-} B(\lambda - T)^{-1}$ *exists and*

$$r_\sigma\left[B(0_- - T)^{-1}\right] < 1$$

then $T + B$ *is the generator of a* bounded (c_0)*-group* $\{V(t); t \in \mathbb{R}\}$. *Moreover, for all* $R \in \mathscr{L}(E)$, $s\lim_{t \to -\infty} RV(t)$ *exists if and only if* $s\lim_{t \to -\infty} RS(t)$ *exists. These limits are then related as*

$$[RV(-\infty)] = [RS(-\infty)]\left[I - B(0_- - T)^{-1}\right]^{-1}.$$

Remark 2. It is known (see [12]) that $T + B$ is the generator of a positive (c_0)-semigroup. We give here a different and *direct* proof which gives us the *additional* and *crucial* information that $\{V(t); t \geq 0\}$ is bounded.

Theorem 1 extends easily to nuclear perturbations of the form $B = \sum B_i$ where B_i is a positive one rank operator with $\sum \|B_i(0_+ - T)^{-1}\| < 1$ (in part (1)) and $\sum \|B_i(0_- - T)^{-1}\| < 1$, in addition (in part (2)).

More precisely we have.

PROPOSITION 1. (1) *Let* $B_i(0_+ - T)^{-1}$ *exist for all* $i \in \mathbb{N}$ *and let* $\sum_{i=0}^{\infty} \|B_i(0_+ - T)^{-1}\| < 1$. *Then* $T + B$ *is the generator of a positive and bounded* (c_0)*-semigroup* $\{V(t); t \geq 0\}$. *Moreover, for all* $R \in \mathscr{L}(E)$, $s\lim_{t \to +\infty} RV(t)$ *exists if and only if* $s\lim_{t \to +\infty} RS(t)$ *exists. These limits are then related as*

$$[RV(+\infty)] = [RS(+\infty)]\left[I - \sum_{i=0}^{\infty} B_i(0_+ - T)^{-1}\right]^{-1}.$$

(2) *If, in addition,* $B_i(0_- - T)^{-1}$ *exists for all* $i \in \mathbb{N}$ *and*

$$\sum_{i=0}^{\infty} \|B_i(0_- - T)^{-1}\| < 1$$

then $T + B$ *is the generator of a bounded* (c_0)*-group* $\{V(t); t \in \mathbb{R}\}$. *Moreover, for all* $R \in \mathscr{L}(E)$, $s\lim_{t\to-\infty} RV(t)$ *exists if and only if* $s\lim_{t\to-\infty} RS(t)$ *exists. These limits are then related as*

$$[RV(-\infty)] = [RS(-\infty)]\left[I - \sum_{0}^{\infty} B_i(0_- - T)^{-1}\right]^{-1}.$$

The proof of Theorem 1 is based on the following lemmas.

LEMMA 1. (1) *Let* $\{W(t); t \geq 0\}$ *be a bounded and positive* (c_0)*-semigroup on* E *with generator* G *and let* $C \in \mathscr{L}(D(G); E)$ *be positive. Then* $\lim_{t\to+\infty} C\int_0^t W(s)x\,ds$ *exists for all* $x \in E$ *if and only if* $\lim_{\lambda\to 0_+} C(\lambda - G)^{-1}x$ *exists for all* $x \in E$. *In such a case we have*

$$C\int_0^{\infty} W(s)x\,ds = C(0_+ - G)^{-1}x \qquad \text{for all } x \in E,$$

where $C(0_+ - G)^{-1}x = \lim_{\lambda\to 0_+} C(\lambda - G)^{-1}x \ \forall x \in E$.

(2) *Similarly if* $\{W(t)\}$ *is a positive and bounded* (c_0)*-group then* $\lim_{t\to-\infty} C\int_t^0 W(s)x\,ds$ *exists for all* $x \in E$ *if and only if* $\lim_{\lambda\to 0_-} C(\lambda - G)^{-1}x$ *exists for all* $x \in E$. *In such a case we have*

$$C\int_{-\infty}^{0} W(s)x\,ds = -C(0_- - G)^{-1}x \qquad \text{for all } x \in E.$$

Proof. Since the positive cone of E is generating we may restrict ourselves to $x \in E_+$. Suppose that the strong limit $s\lim_{t\to+\infty} C\int_0^t W(s)\,ds$ exists. We denote this strong limit by $C\int_0^\infty W(s)ds$. Then

$$C\int_0^{\infty} W(s)x\,ds \geq C\int_0^{\infty} e^{-\lambda s}W(s)x\,ds = C(\lambda - G)^{-1}x$$

for all $x \in E_+$ and $\lambda > 0$. Since $(\lambda - G)^{-1}$ is nonincreasing for $\lambda > 0$ and E is an order continuous norm, it follows that $\lim_{\lambda\to 0^+} C(\lambda - G)^{-1}x$ exists in E and

$$C(0_+ - G)^{-1} \leq C\int_0^{\infty} W(s)\,ds.$$

Conversely suppose that $C(0_+ - G)^{-1}$ exists in E. From the inequality

$$C(0_+ - G)^{-1} \geq C(\lambda - G)^{-1} \qquad \text{for all } \lambda > 0$$

we deduce

$$C\int_0^t e^{-\lambda s}W(s)x\,ds \leq C(\lambda - G)^{-1}x \leq C(0_+ - G)^{-1}x$$

for all $x \in E_+$, $0 < t < \infty$, and $\lambda > 0$. Letting $\lambda \to 0_+$ we get

$$C\int_0^t W(s)x\,ds \leq C(0_+ - G)^{-1}x \qquad \text{for all } x \in E_+, t < +\infty.$$

Now letting $t \to +\infty$ we get the reverse inequality

$$C\int_0^\infty W(s)\,ds \leq C(0_+ - G)^{-1}.$$

This proves the first part of Lemma 1.

Let $\tilde{W}(t) = W(-t)(t \geq 0)$, $\{\tilde{W}(t), t \geq 0\}$ is a positive bounded (c_0)-semigroup with generator $\tilde{G} = -G$ so that, according to the preceding analysis $C\int_0^\infty \tilde{W}(s)\,ds$ exists if and only if $C(0_+ - \tilde{G})^{-1}$ exists and these strong limits are equal. This amounts to $C\int_{-\infty}^0 W(s)\,ds = -C(0_- - G)^{-1}$. ∎

Now we introduce the space H_∞^c of strongly continuous mappings

$$Z: t \in [0, \infty) \to Z(t) \in \mathscr{L}(E)$$

such that $\sup_{t \geq 0}\|Z(t)\| < +\infty$ endowed with the canonical norm

$$|||Z||| = \sup_{t \geq 0}\|Z(t)\|, \qquad Z \in H_\infty^c.$$

Let $\widetilde{H_\infty^c}$ be the (closed) subspace of H_∞^c consisting of those elements having a strong limit as $t \to +\infty$. Finally we define on H_∞^c the operator

$$L: Z \in H_\infty^c \to \int_0^t Z(s)BS(t-s)\,ds,$$

where the integral is understood as a strong one.

LEMMA 2. *The operator L is bounded on H_∞^c. Moreover*

$$\|L^n\|_{\mathscr{L}(H_\infty^c)} \leq \left\|\left[B(0_+ - T)^{-1}\right]^n\right\| \qquad \forall n \in \mathbb{N}.$$

In particular $r_\sigma(L) \leq r_\sigma[B(0_+ - T)^{-1}]$. Finally L leaves invariant $\widetilde{H_\infty^c}$ and

$$s\lim_{t \to +\infty} LZ(t) = \left[s\lim_{t \to +\infty} Z(t)\right]B(0_+ - T)^{-1}.$$

Proof. Let $x \in D(T)$ and $Z \in H_\infty^c$

$$LZ(t)x = \int_0^t Z(s)BS(t-s)x\,ds,$$

so that

$$\begin{aligned}\|LZ(t)x\| &\le \int_0^t \|Z(s)\|\,\|BS(t-s)x\|\,ds\\ &\le |||Z|||\int_0^t \|BS(t-s)x\|\,ds\\ &\le |||Z|||\int_0^\infty \|BS(s)x\|\,ds\end{aligned}$$

and

$$\begin{aligned}\|BS(s)x\| &= \|\varphi(S(s)x_+)b - \varphi(S(s)x_-)b\|\\ &\le \big(\varphi(S(s)x_+) + \varphi(S(s)x_-)\big)\|b\|\\ &= (\varphi(S(s)|x|))\|b\|\end{aligned}$$

give

$$\begin{aligned}\|LZ(t)x\| &\le |||Z|||\int_0^\infty \varphi(S(s)|x|)\|b\|\,ds\\ &= |||Z|||\left\|\int_0^\infty \varphi(S(s)|x|)b\,ds\right\|\\ &= |||Z|||\left\|\int_0^\infty BS(s)|x|\,ds\right\|.\end{aligned}$$

Using Lemma 1 we deduce

$$\begin{aligned}\|LZ(t)x\| &\le |||Z|||\left\|B(0_+ - T)^{-1}|x|\right\|\\ &\le |||Z|||\left\|B(0_+ - T)^{-1}\right\|\|x\|.\end{aligned}$$

In view of denseness of $D(T)$ in E we have

$$\|LZ(t)x\| \le |||Z|||\left\|B(0_+ - T)^{-1}\right\|\|x\| \qquad \forall x \in E,$$

so that

$$\|LZ(t)\| \le |||Z|||\left\|B(0_+ - T)^{-1}\right\| \qquad \forall t \ge 0$$

and then

$$|||LZ||| \le |||Z|||\left\|B(0_+ - T)^{-1}\right\| \qquad \forall Z \in H_\infty^c.$$

Finally

$$\|L\|_{\mathscr{L}(H_\infty^c)} \le \left\|B(0_+ - T)^{-1}\right\|.$$

To prove that LZ is strongly continuous we use density arguments: If $x \in D(T)$ then $LZ(t)x$ is continuous.

Let $x \in E$ and $\{x_n\} \subset D(t)$ be such that $x_n \to x$ in E

$$\| LZ(t)x_n - LZ(t)x_m \| \leq \int_0^t \| Z(s) \| \, \| BS(t-s)(x_n - x_m) \| \, ds.$$

We can obtain, as previously, the estimate

$$\| LZ(t)x_n - LZ(t)x_m \| \leq \| Z \| \left\| B(0_+ - T)^{-1} \right\| \|x_n - x_m\|.$$

This shows that $\{LZ(t)x_n\}$ is a cauchy sequence in E uniformly in $t \geq 0$, whence $LZ(t)x_n \to LZ(t)x$, $n \to \infty$ uniformly in $t \geq 0$ so that LZ is strongly continuous.

A computation shows that

$$\begin{aligned} L^n Z(t) = \int_0^t dt_n \int_0^{t_n} dt_{n-1} \ldots \int_0^{t_2} dt_1 Z(t_1) BS(t_2 - t_1) \ldots \\ BS(t_n - t_{n-1}) BS(t - t_n). \end{aligned}$$

Hence, for $x \in D(T)$

$$\begin{aligned} \| L^n Z(t)x \| &\leq \int_0^t dt_n \ldots \int_0^{t_2} dt_1 \| Z(t_1) \| \, \| BS(t_2 - t_1) \ldots \\ &\qquad \times BS(t_n - t_{n-1}) BS(t - t_n)|x| \| \\ &\leq \| Z \| \int_0^t dt_n \ldots \int_0^{t_2} dt_1 \varphi\big(S(t_2 - t_1)b\big) \ldots \\ &\qquad \varphi\big(S(t_n - t_{n-1})b\big) \varphi\big(S(t - t_n)|x|\big) \|b\| \\ &= \| Z \| \left\| \int_0^t dt_n \ldots \int_0^{t_2} dt_1 \varphi\big(S(t_2 - t_1)b\big) \ldots \right. \\ &\qquad \left. \varphi\big(S(t_n - t_{n-1})b\big) \varphi\big(S(t - t_n)|x|\big) b \right\| \\ &= \| Z \| \left\| \int_0^t dt_n \ldots \int_0^{t_2} dt_1 BS(t_2 - t_1) \ldots \right. \\ &\qquad \left. BS(t_n - t_{n-1}) BS(t - t_n)|x| \right\|. \end{aligned}$$

The change of variables

$$t - t_n = u_n$$
$$t_n - t_{n-1} = u_{n-1}$$
$$\vdots$$
$$t_2 - t_1 = u_1$$

shows that

$$\begin{aligned}
\| L^n Z(t) x \| &\leq |||\, Z \,||| \left\| \int_{u_1 + \dots u_n \leq t} BS(u_1) \dots BS(u_n)\, du_1 \dots du_n |x| \right\| \\
&\leq |||\, Z \,||| \left\| \int_{u_i \geq 0} BS(u_1) \dots BS(u_n)\, du_1 \dots du_n |x| \right\| \\
&= |||\, Z \,||| \left\| \left[B(0_+ - T)^{-1} \right]^n |x| \right\| \\
&\leq |||\, Z \,||| \left\| \left[B(0_+ - T)^{-1} \right]^n \right\| \|x\|.
\end{aligned}$$

As previously, the latter estimate remains true for all $x \in E$ so that

$$\| L^n Z(t) \| \leq |||\, Z \,||| \left\| \left[B(0_+ - T)^{-1} \right]^n \right\| \qquad \forall t \geq 0$$

and then

$$|||\, L^n Z \,||| \leq |||\, Z \,||| \left\| \left[B(0_+ - T)^{-1} \right]^n \right\| \qquad \forall Z \in H_\infty^c$$

finally

$$\|L^n\|_{\mathscr{L}(H_\infty^c)} \leq \left\| \left[B(0_+ - T)^{-1} \right]^n \right\| \qquad \forall n \in \mathbb{N}$$

letting $n \to \infty$ we get

$$r_\sigma(L) \leq r_\sigma\left[B(0_+ - T)^{-1} \right].$$

Suppose now that $Z \in \widetilde{H_\infty^c}$, i.e., $\lim_{t \to +\infty} Z(t)x$ exists for all $x \in E$. Then for $x \in D(T)$, we have

$$\begin{aligned}
LZ(t)x &= \int_0^t Z(s) BS(t-s) x\, ds \\
&= \int_0^t Z(t-s) BS(s) x\, ds \\
&= \int_0^\infty 1_{[0,t]}(s) Z(t-s) Bs(s) x\, ds.
\end{aligned}$$

The dominated convergence theorem yields

$$\lim_{t \to +\infty} LZ(t)x = Z(+\infty)\int_0^\infty BS(s)x\,ds$$
$$= s \lim_{t \to +\infty} Z(t)\left[B(0_+ - T)^{-1}\right]x.$$

Finally a density argument ends the proof. ∎

We are now ready to prove Theorem 1.

Proof of Theorem 1. In view of Lemma 1, since $S(\cdot) \in H_\infty^c$ and $r_\sigma(L) \leq r_\sigma[B(0_+ - T)^{-1}] < 1$, there exists a unique $V(\cdot) \in H_\infty^c$ such that

$$V - LV = S \qquad \text{in } H_\infty^c$$

given by

$$V = \sum_0^\infty L^n S.$$

One can verify (see, for instance, [14, Proof of Lemma 3]) that $V(\cdot)$ has the semigroup property and that its generator is $T + B$. Let $R \in \mathscr{L}(E)$ and we have

$$[RV] - L[RV] = RS.$$

Therefore $RV \in \widetilde{H_\infty^c}$ if and only if $RS \in \widetilde{H_\infty^c}$ because L leaves invariant $\widetilde{H_\infty^c}$. Moreover, according to Lemma 2,

$$[RV](+\infty) - [RV](+\infty)B(0_+ - T)^{-1} = [RS](+\infty).$$

Consider now the second part of Theorem 1. To prove that $T + B$ generates a bounded group it suffices to prove that $-T - B$ generates a bounded semigroup. Let $\tilde{S}(t) = S(-t)$, $t \geq 0$. Consider the Fredholm equation

$$\tilde{V} = \tilde{S} - \tilde{L}\tilde{V} \qquad \text{in } H_\infty^c,$$

where $\tilde{L}$: $Z \in H_\infty^c \to \int_0^t Z(s)B\tilde{S}(t-s)\,ds$. One can prove, as in Lemma 2, that

$$\tilde{L} \in \mathscr{L}(H_\infty^c) \qquad \text{and} \qquad r_\sigma(\tilde{L}) \leq r_\sigma\left[B(0_- - T)^{-1}\right].$$

Hence we follow the same strategy as in the first part. ∎

Proof of Proposition 1. We define H_∞^c, $\widetilde{H_\infty^c}$, L, and $\tilde{L}$ as in the proof of Theorem 1 and we show similarly that

$$\|L\|_{\mathscr{L}(H_\infty^c)} \leq \sum_{i=0}^{\infty} \left\| \left[B_i(0_+ - T)^{-1} \right] \right\|$$

and

$$\|\tilde{L}\|_{\mathscr{L}(H_\infty^c)} \leq \sum_{i=0}^{\infty} \left\| \left[Bi(0_- - T)^{-1} \right] \right\|.$$

Then we argue as in the proof of Theorem 1. ∎

Remark 3. We were unable to give simple criteria for a perturbation B to have the above decomposition.

III. EXISTENCE OF WAVE OPERATORS

In this section we will show how the existence of wave operators is closely related to the existence of strong limits $B(0_+ - T)^{-1}$, $B(0_- - T)^{-1}$ and to their size. More precisely we have the following

THEOREM 2. *If both* $B(0_+ - T)^{-1}$ *and* $B(0_- - T)^{-1}$ *exist and if* $r_\sigma[B(0_+ - T)^{-1}] < 1$ *then* $s\lim_{t\to+\infty} V(t)S(-t)$ *exists.*

Proof. According to Theorem 1 (part (1)), $T + B$ generates a bounded semigroup $\{V(t); t \geq 0\}$ and

$$V(t) = S(t) + \int_0^t V(s)BS(t-s)\,ds$$

so that

$$V(t)S(-t)x = x + \int_0^t V(s)BS(-s)x\,ds.$$

On the other hand, for $x \in D(T) \cap E_+$,

$$\begin{aligned}
\int_0^t \|V(s)BS(-s)x\|\,ds &\le |||V|||\int_0^\infty \|BS(-s)x\|\,ds \\
&= |||V|||\int_{-\infty}^0 \|BS(s)x\|\,ds \\
&= |||V|||\int_{-\infty}^0 (\varphi(S(s)x))\|b\|\,ds \\
&= |||V|||\left\|\int_{-\infty}^0 BS(s)x\,ds\right\| \\
&= |||V|||\left\|-B(0_- - T)^{-1}x\right\| \\
&\le |||V|||\left\|B(0_- - T)^{-1}\right\|\|x\|.
\end{aligned}$$

Hence, by density arguments, $\lim_{t\to+\infty} V(t)S(-t)x$ exists for all $x \in E_+$, and consequently, $s\lim_{t\to+\infty} V(t)S(-t)$ exists. ∎

Symmetrically we have the following

THEOREM 3. *Let both $B(0_+ - T)^{-1}$ and $B(0_- - T)^{-1}$ exist and let $r_\sigma[B(0_- - T)^{-1}] < 1$, then $s\lim_{t\to-\infty} V(t)S(-t)$ exists.*

Proof. According to Theorem 1 (part (2)), $-T - B$ generates a bounded (c_0)-semigroup $\{\tilde{V}(t); t \ge 0\}$ such that

$$\tilde{V}(t) = \tilde{S}(t) - \int_0^t \tilde{V}(s)B\tilde{S}(t-s)\,ds, \qquad t \ge 0,$$

where $\tilde{S}(t) = S(-t)$, i.e.,

$$\tilde{V}(t) = S(-t) - \int_0^t \tilde{V}(s)BS(-t+s)\,ds$$

so that

$$\tilde{V}(t)S(t)x = x - \int_0^t \tilde{V}(s)BS(s)x\,ds.$$

On the other hand, for $x \in D(T) \cap E_+$,

$$\begin{aligned}\int_0^t \left\| \tilde{V}(s) BS(-s) x \right\| ds &\leq \||\tilde{V}\|| \int_0^\infty \| BS(s) x \| ds \\ &= \||\tilde{V}\|| \int_0^\infty (\varphi(S(s)x)) \|b\| \, ds \\ &= \||\tilde{V}\|| \left\| \int_0^\infty BS(s) x \, ds \right\| \\ &= \||\tilde{V}\|| \left\| B(0_+ - T)^{-1} x \right\| \\ &\leq \||\tilde{V}\|| \left\| B(0_+ - T)^{-1} \right\| \|x\|. \end{aligned}$$

Hence by density arguments, $s \lim_{t \to +\infty} \tilde{V}(t) S(t)$ exists. This ends the proof since by definition $V(t) = \tilde{V}(-t)$, $t \leq 0$. ∎

To prove the existence of $s \lim_{t \to +\infty} S(-t)V(t)$ we need the following

LEMMA 3. *If $B(0_+ - T)^{-1}$ exists and $r_\sigma[B(0_+ - T)^{-1}] < 1$ then* $(0, \infty) \subset \varrho(A)$, $(\lambda - A)^{-1}$ *is positive* $(\lambda > 0)$, *and* $B(0_+ - A)^{-1}$ *exists* (*where* $A = T + B$).

Proof. See in [1] the proof of Lemma 3 (and use the order continuous property of the norm). ∎

With the above lemma we prove the following

THEOREM 4. *Let $B(0_+ - T)^{-1}$ exist and let $r_\sigma[B(0_+ - T)^{-1}] < 1$, then $s \lim_{t \to +\infty} S(-t)V(t)$ exists.*

Proof. According to Lemma 3, $B(0_+ - A)^{-1}$ exists or, equivalently, $\int_0^\infty BV(s)\, ds$ exists. Then from

$$V(t) = S(t) + \int_0^t S(t-s) BV(s) \, ds$$

we deduce

$$S(-t)V(t)x = x + \int_0^t S(-s) BV(s) x \, ds.$$

On the other hand, for $x \in D(T) \cap E_+$

$$\begin{aligned}\int_0^t \| S(-s) BV(s) x \| \, ds &\le c \int_0^t \| BV(s) x \| \, ds \\ &\le c \int_0^\infty \| BV(s) x \| \, ds \\ &= c \int_0^\infty \varphi(V(s)x) \|b\| \, ds \\ &= c \left\| \int_0^\infty BV(s) x \, ds \right\| \\ &= c \left\| B(0_+ - A)^{-1} x \right\| \\ &\le c \left\| B(0_+ - A)^{-1} \right\| \|x\|,\end{aligned}$$

where $c = \sup_{t \le 0} \|S(-t)\|$. This ends our proof. ∎

It is well known that the existence of wave operators implies the similarity on T and $T + B$ (see [10, Chap. X]). Hence

THEOREM 5. *Let both $B(0_+ - T)^{-1}$ and $B(0_- - T)^{-1}$ exist with $r_\sigma[B(0_+ - T)^{-1}] < 1$ and $r_\sigma[B(0_- - T)^{-1}] < 1$ then $W_+(T, T + B) = s\lim_{t \to +\infty} S(-t)V(t)$ and $W_+(T + B, T) = s\lim_{t \to +\infty} V(-t)S(t)$ exist. Moreover*

$$W_+(T + B, T) W_+(T, T + B) = W_+(T, T + B) W_+(T + B, T) = I$$

and

$$T + B = [W_+(T + B, T)] T [W_+(T + B, T)]^{-1}.$$

IV. ON THE BOUNDEDNESS OF HEAT SEMIGROUPS WITH POSITIVE COMPACT POTENTIALS

Theorem 1 provides us with a criterion of boundedness of perturbed semigroups on Banach lattices in terms of limiting absorption principles. The aim of this section is to exemplify this result by applying it to heat semigroups on $\mathbb{R}^N$.

Let $T = \Delta$ be the laplacian operator on $L^p(\mathbb{R}^N)$ which generates the positive semigroup of contractions

$$T(t) f(y) = (4\pi t)^{-N/2} \int \exp\left(-|y - z|^2 / 4t\right) f(z) \, dz,$$

where $N \geq 3$ and $1 < p < \infty$. We define a rank one operator from $W^{2,p}$ to L^p by

$$Bf = \varphi(f)g, \quad \text{where } g \in L^p_+(\mathbb{R}^N)$$

and

$$\varphi \in D(\Delta)' \subset \mathscr{D}'(\mathbb{R}^N), \ \varphi \geq 0.$$

As a *positive* distribution φ extends to a *measure* $d\mu$, then we have

THEOREM 6. $B(0_+ - \Delta)^{-1}$ *exists if and only if* $\int (d\mu(x)/|x-y|^{N-2}) \in L^{p^*}(\mathbb{R}^N)$ (*where* $1/p + 1/p^* = 1$).

Proof. Let $f \in L^p_+$ and $\lambda > 0$ then

$$B(\lambda - \Delta)^{-1} f(x) = g(x) \int d\mu(y) \int_0^\infty \frac{e^{-\lambda t}}{(4\pi t)^{N/2}} \int e^{-|y-z|^2/4t} f(z)\, dz\, dt.$$

By using the monotone convergence theorem

$$\lim_{\lambda \to 0_+} B(\lambda - \Delta)^{-1} f(x)$$
$$= g(x) \int d\mu(y) \int_0^\infty \frac{1}{(4\pi t)^{N/2}} \int e^{-|y-z|^2/4t} f(z)\, dz\, dt.$$

On the other hand, making a change of variable and using the Fubini Theorem, we get

$$\int d\mu(y) \int_0^\infty \frac{1}{(4\pi t)^{N/2}} \int e^{-|y-z|^2/4t} f(z)\, dz\, dt$$
$$= \frac{\Gamma(N/2 - 1)}{4\pi^{N/2}} \iint \frac{f(z)}{|y-z|^{N-2}}\, d\mu(y)\, dz.$$

Now let $\int (d\mu(y)/|y-z|^{N-2}) \in L^{p^*}(\mathbb{R}^N)$. Then the strong limit $s \lim_{\lambda \to 0_+} B(\lambda - \Delta)^{-1}$ exists and

$$\left\| B(0_+ - \Delta)^{-1} \right\| \leq \frac{\Gamma(N/2 - 1) \|g\|_p}{4\pi^{N/2}} \left\| \int \frac{d\mu(x)}{|x-y|^{N-2}} \right\|_{p^*}.$$

Conversely, if $B(0_+ - \Delta)^{-1}$ exists then the previous equalities give

$$\frac{\Gamma(N/2 - 1) \|g\|_p}{4\pi^{N/2}} \iint \frac{f(z)}{|y-z|^{N-2}} d\mu(y) dz = \left\| B(0_+ - \Delta)^{-1} f \right\|$$

so that the left hand side of the above inequality defines a continuous functional on L^p and therefore $\int (d\mu(y)/|x-y|^{N-2}) \in L^{p^*}(\mathbb{R}^N)$. ∎

We introduce the space $\mathscr{C}$ of measures $d\mu$ such that $\int (d\mu(x)/|x-y|^{N-2}) \in L^{p^*}(\mathbb{R}^N)$ endowed with the halfnorm $|||\, d\mu \,||| = \| \int (d\mu(x)/|x-y|^{N-2})\|_{p^*}$ and let $\mathscr{C}_+$ be the positive cone of $\mathscr{C}$. Then we have the following

THEOREM 7. *Let $d\mu \in \mathscr{C}_+$, then $B(0_+ - \Delta)^{-1}$ exists and*

$$r = r_\sigma\left[B(0_+ - \Delta)^{-1}\right] = \frac{\Gamma(N/2-1)}{4\pi^{N/2}} \iint \frac{g(x)}{|x-y|^{N-2}}\, dx\, d\mu(y).$$

Consequently if

$$\iint \frac{g(x)}{|x-y|^{N-2}}\, dx\, d\mu(y) < \frac{4\pi^{N/2}}{\Gamma(N/2-1)}$$

then $\{e^{t(\Delta+B)}; t \geq 0\}$ is a bounded semigroup and $e^{t(\Delta+B)}$ tends to 0 strongly in L^p as $t \to +\infty$.

Proof. By Theorem 6, $B(0_+ - \Delta)^{-1}$ exists and is a positive compact (one rank) operator so that r is an *eigenvalue* of $B(0_+ - \Delta)^{-1}$. Let $f \in L^p$ be such that $B(0_+ - \Delta)^{-1} f = rf$, $f \geq 0$, i.e.,

$$\frac{\Gamma(N/2-1)\, g(x)}{4\pi^{N/2}} \iint \frac{f(z)}{|y-z|^{N-2}} d\mu(y)\, dz = rf(x)$$

then

$$\frac{\Gamma(N/2-1)}{4\pi^{N/2}} \iint \frac{g(x)}{|x-y|^{N-2}}\, dx\, d\mu(y) \iint \frac{f(z)}{|y-z|^{N-2}}\, d\mu(y)\, dz$$
$$= r \iint \frac{f(x)}{|x-y|^{N-2}}\, dx\, d\mu(y)$$

and

$$r = \frac{\Gamma(N/2-1)}{4\pi^{N/2}} \iint \frac{g(x)}{|x-y|^{N-2}}\, dx\, d\mu(y).$$

Finally we apply Theorem 1 (part (1)), since $r_\sigma[B(0_+ - \Delta)^{-1}] < 1$ and $e^{t\Delta} \to 0$ strongly as $t \to \infty$ (see, for instance, [13, p. 110]). ∎

Similarly we have the following

THEOREM 8. *Let* $B = \sum_{i=0}^{\infty}(\ ,d\mu_i)g_i$ *where* $g_i \in L^p$, $\|g_i\|_p = 1$, *and* $d\mu_i \in \mathscr{C}_+$. *Then* $B_i(0_+ - \Delta)^{-1}$ *exists for* $i = 1, 2, \ldots$ *and*

$$\left\| B_i(0_+ - \Delta)^{-1} \right\| \le \frac{\Gamma(N/2-1)}{4\pi^{N/2}} |||\, d\mu_i \,||| .$$

Consequently, if $(|||\, d\mu_i \,|||) \in l^1$ *and*

$$\sum_{i=0}^{\infty} |||\, d\mu_i \,||| < \frac{4\pi^{N/2}}{\Gamma(N/2-1)}$$

then $\Delta + B$ *generates a bounded and positive* (c_0)*-semigroup* $\{e^{t(\Delta+B)}; t \ge 0\}$ *and* $e^{t(\Delta+B)}$ *tends to* 0 *strongly in* L^p *as* $t \to +\infty$.

Proof. In view of Theorem 6, $B_i(0_+ - \Delta)^{-1}$ exists and

$$\left\| B_i(0_+ - \Delta)^{-1} \right\| \le \frac{\Gamma(N/2-1)}{4\pi^{N/2}} |||\, d\mu_i \,|||$$

gives

$$\sum_{i=0}^{\infty} \left\| B_i(0_+ - \Delta)^{-1} \right\| \le \frac{\Gamma(N/2-1)}{4\pi^{N/2}} \sum_{i=0}^{\infty} |||\, d\mu_i \,||| < 1.$$

Then the result is a simple consequence of Proposition 1. ∎

If $d\mu$ is a function, i.e., $d\mu(x) = h(x)dx$ with $h \in L^1_{Loc}(\mathbb{R}^N)$ then we have the following

THEOREM 9. *Let* $N \ge 3$, $p < N/2$, *and* $h \in L^r \cap L^q$ *with* $1 \le q < Np/(Np + 2p - N) < r < \infty$, *then* $B \in \mathscr{L}(W^{2,p}; L^p)$; $B(0_+ - \Delta)^{-1}$ *exists and*

$$\left\| B(0_+ - \Delta)^{-1} \right\| \le \frac{\Gamma(N/2-1)}{4\pi^{N/2}} \left(c_r \left(\frac{-d_q c_q}{d_r c_r} \right)^{d_r/(d_r - d_q)} + c_q \left(\frac{-d_q c_q}{d_r c_r} \right)^{d_q/(d_r - d_q)} \right) \|g\|_p ,$$

where $c_i = \|h\|_i(mes(S_{N-1})(2pi - i - p)/|Npi + 2pi - pN - Ni|)$ *and* $d_i = 2 + N - N/p - N/i$ $(i = r, q)$. *Consequently if*

$$\|g\|_p \left(c_r \left(\frac{-d_q c_q}{d_r c_r} \right)^{d_r/(d_r - d_q)} + c_q \left(\frac{-d_q c_q}{d_r c_r} \right)^{d_q/(d_r - d_q)} \right) < \frac{4\pi^{N/2}}{\Gamma(N/2-1)}$$

then $\Delta + B$ *generates a bounded* (c_0)*-semigroup of positive operators. Moreover* $e^{t(\Delta+B)} \to 0$ *strongly as* $t \to \infty$.

Proof. In view of Theorem 6 it suffices to prove that $\int (d\mu(y)/|y-z|^{N-1}) = \int (h(y)\,dy/|y-z|^{N-2}) \in L^{p^*}(\mathbb{R}^N)$. Let $R > 0$ and let

$$\begin{aligned}\tilde{h}(z) &= \int \frac{h(y)}{|y-z|^{N-2}}\,dy \\ &= \int_{|y-z|<R} \frac{h(y)}{|y-z|^{N-2}}\,dy + \int_{|y-z|>R} \frac{h(y)}{|y-z|^{N-2}}\,dy \\ &= (h*k)(z) + (h*l)(z),\end{aligned}$$

where $(k(u) = \chi_{\{|u|<R\}})/|u|^{N-2}$ and $(l(u) = \chi_{\{|u|>R\}}/|u|^{N-2}$. On the other hand $k \in L^s$ with $1/p^* = 1/r + 1/s - 1$ and by Young's theorem

$$\begin{aligned}\|h*k\|_{p^*} &\le \|h\|_r \|k\|_s \\ &= c_r R^{d_r},\end{aligned}$$

where $c_r = \|h\|_r((mes(S_{N-1})(2pr - r - p))/(Npr + 2pr - pN - Nr))$ and $d_r = 2 + N - N/p - N/r$. The same arguments imply that $h*l \in L^{p^*}$ and $\|h*k\|_{p^*} \le c_q R^{d_q}$ where $c_q = \|h\|_q((mes(S_{N-1})(2pq - q - p))/|Npq + 2pq - pN - Nq|)$ and $d_q = 2 + N - N/p - N/q$. Hence $\tilde{h} \in L^{p^*}$ and $\|\tilde{h}\|_{p^*} \le c_r R^{d_r} + c_q R^{d_q}$. Let $R = (-c_q d_q/c_r d_r)^{1/(d_r - d_q)}$ (the optimal estimation), then $B(0_+ - \Delta)^{-1}$ exists and

$$\begin{aligned}\left\| B(0_+ - \Delta)^{-1} f \right\| \le{}& \frac{\|g\|_p \Gamma(N/2 - 1)\|f\|_p}{4\pi^{N/2}} \\ &\times \left(c_r \left(\frac{-d_q c_q}{d_r c_r} \right)^{d_r/(d_r - d_q)} + c_q \left(\frac{-d_q c_q}{d_r c_r} \right)^{d_q/(d_r - d_q)} \right).\end{aligned}$$

Finally Theorem 1 (part (1)) ends the proof. ∎

Remark 4. In $L^1(\mathbb{R}^N)$, $e^{t\Delta}$ is *isometric* on the positive cone. Then, according to [1] the boundedness of $e^{t(\Delta+B)}$ is *equivalent* to $r_\sigma[B(0_+ - \Delta)^{-1}] < 1$ so we have the following *optimal* result

PROPOSITION 2. *Let* $p = 1$, $N \ge 3$, *and let* $d\mu$ *be a positive measure such that*

$$\int \frac{d\mu(x)}{|x-y|^{N-2}} \in L^\infty.$$

Then $\Delta + B$ *generates a bounded semigroup* $\{e^{t(\Delta+B)}; t \geq 0\}$ *if and only if* $\int\int(g(x)/|x-y|^{N-2})dx\,d\mu(y) < 4\pi^{N/2}/\Gamma(N/2-1)$.

Proof. In view of Theorem 7, $B(0_+ - \Delta)^{-1}$ exists and

$$r = r_\sigma\left[B(0_+ - \Delta)^{-1}\right] = \frac{\Gamma(N/2-1)}{4\pi^{N/2}} \int\int \frac{g(x)}{|x-y|^{N-2}}\,dx\,d\mu(y).$$

Then the result is a consequence of Corollary 2 of Theorem 6 in [1]. ∎

We close this section by the following observations when $N = 1, 2$.

Remark 5. Let $N = 1, 2$ and let $B \in \mathscr{L}(L^1(\mathbb{R}^N))$ be a positive one rank operator. Then the strong limit of $B(\lambda - \Delta)^{-1}$ as $\lambda \to 0_+$ does not exist and consequently, (see Remark 4), $\{e^{t(\Delta+B)}; t \geq 0\}$ *is not bounded*. We conjecture that this property remains true in L^p spaces $(1 < p < +\infty)$.

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